A Deflated Iterative Solver for Magnetostatic Finite Element Models with Large Differences in Permeability

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***Résumé* - La présence de matériaux dont les perméabilités sont très différents a une influence néfaste sur la convergence des solveurs itératifs de Krylov. Le préconditionnement ne peut pas améliorer la convergence de certains éléments, dont les vecteurs propres correspondent aux domaines où les matériaux présentent une faible perméabilité. Des approximations pour ces vecteurs propres sont réalisées sur base des caractéristiques physiques du problème. La procédure itérative de résolution est séparée en un problème réduit pour les modes propres de faible convergence, et un modèle complet dont ces mêmes modes sont retirés. Cette méthode converge plus rapidement que les approches conventionnelles.**

***Abstract* - The presence of materials with a large relative difference in permeability has a harmful influence on the convergence of Krylov subspace iterative solvers. Some slow converging components are not cured by preconditioning and correspond to eigenvectors reflecting the domains with relatively low permeable material. Approximations for those eigenvectors are determined using physical knowledge of the problem. The iterative solution process is split up in a small problem counting for the separated eigenmodes and a full-size problem out of which the slow converging modes are removed. This deflated preconditioned solver is faster converging compared to more common approaches.**

I. CONVERGENCE OF ICCG

Krylov subspace iterative methods solving linear systems of equations require only matrix-vector multiplications and vector updates, making them especially attractive to solve the sparse systems arising from finite element discretisations [1]. The magnetostatic Poisson equation in terms of the magnetic vector potential, discretised by finite elements, yields a sparse positive definite symmetric system of equations, $\mathbf{Ax} = \mathbf{b}$, to which the Conjugate Gradient (CG) method is applicable.

The convergence of CG applied to (1) is bound by

$$\left\|\mathbf{x} - \mathbf{x}^{(k)}\right\|_{\mathbf{A}} \leq 2\left\|\mathbf{x} - \mathbf{x}^{(0)}\right\|_{\mathbf{A}} \left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^{k} . \qquad (1)$$

The condition number K is the ratio between the largest and the smallest eigenvalue [1]. k is the iteration number. The convergence of CG applied to the model problem of Fig. 1 is plotted in Fig. 2. Better convergence is achieved by applying the Krylov subspace method to the system

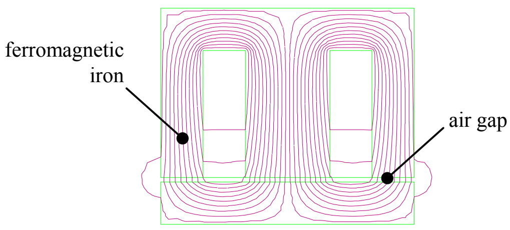

Fig. 1: Magnetic flux plot of an inductor.

$$\mathbf{M}^{-1}\mathbf{Ax} = \mathbf{M}^{-1}\mathbf{b} \qquad (2)$$

with $\mathbf{M}$ an appropriate preconditioner. A good preconditioner projects the spectrum of $\mathbf{A}$ to a spectrum for $\mathbf{M}^{-1}\mathbf{A}$ with all eigenvalues in a small band around 1, diminishing K and thus increasing the convergence rate.

As a preconditioner, an Incomplete Cholesky (IC) factorisation is commonly used [2]. The spectra of $\mathbf{A}$ and $\mathbf{M}^{-1}\mathbf{A}$ are plotted in Fig. 3. Preconditioning improves the convergence substantially (Fig. 2, Table I). The two smallest, but important, eigenvalues left after preconditioning, λ_1 and λ_2, are related to the presence of two relatively low permeable air parts inside the model. The iterative solver only reaches the solution when both eigenmodes are

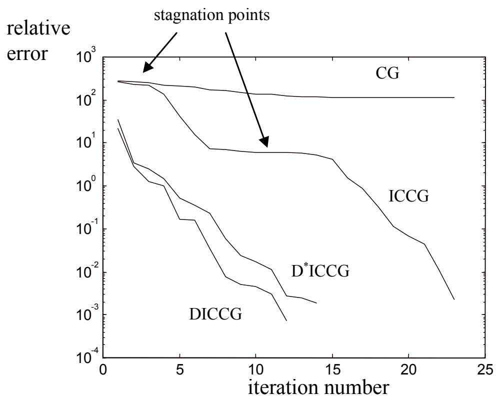

Fig. 2: Convergences of CG, ICCG, exact deflated ICCG (DICCG) and approximative deflated ICCG (D*ICCG).

TABLE I
NUMBER OF ITERATIONS

number of unknowns	CG	ICCG	D*ICCG	DICCG
117	111	23	14	12
424	239	37	31	21

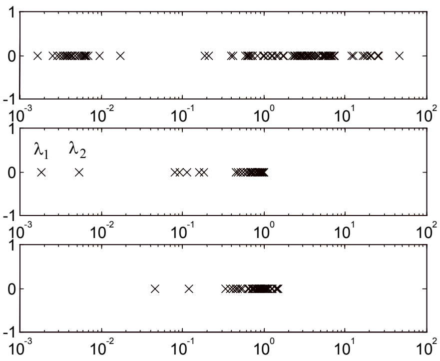

Fig. 3: Spectra of **A** (above), $\mathbf{M}^{-1}\mathbf{A}$ (middle) and $\mathbf{M}^{-1}\mathbf{P}^{T}\mathbf{A}$ (under).

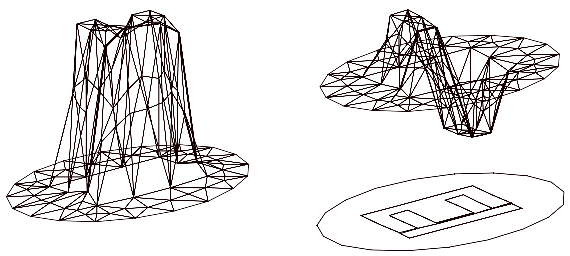

Fig. 4: Eigenvectors of $\mathbf{M}^{-1}\mathbf{A}$ corresponding to λ_1 and λ_2.

incorporated. The two stagnation points in the convergence history (Fig. 2) reveal that these eigenmodes are difficult to find.

II. Deflated ICCG

The eigenvectors associated with λ_1 and λ_2 (Fig. 4), form a partial eigenspace $\mathbf{V} = [\mathbf{v}_1 \quad \mathbf{v}_2]$. Consider the operator

$$\mathbf{P} = \mathbf{I} - \mathbf{V}\mathbf{E}^{-1}(\mathbf{A}\mathbf{V})^{T} \tag{3}$$

with $\mathbf{E} = (\mathbf{A}\mathbf{V})^{T}\mathbf{V}$. **P** is a projector ($\mathbf{P}^2 = \mathbf{P}$) and commutates with **A** as $\mathbf{P}^{T}\mathbf{A} = \mathbf{A}\mathbf{P}$ [3]. The solution **x** is split up in

$$\mathbf{x} = (\mathbf{I} - \mathbf{P})\mathbf{x} + \mathbf{P}\mathbf{x}\,. \tag{4}$$

$(\mathbf{I} - \mathbf{P})\mathbf{x}$ is the component of the solution contained in the low dimensional space spanned by **V**. As a consequence, its computation is inexpensive.

$$(\mathbf{I} - \mathbf{P})\mathbf{x} = \mathbf{V}\mathbf{E}^{-1}\mathbf{V}^{\mathbf{T}}\mathbf{b}\,. \tag{5}$$

$\mathbf{P}\mathbf{x}$ is perpendicular to **V** in the $\mathbf{M}^{-1}\mathbf{A}$-inner product. This second component is solved from

$$\mathbf{M}^{-1}\mathbf{P}^{T}\mathbf{A}\mathbf{x} = \mathbf{M}^{-1}\mathbf{P}^{T}\mathbf{b}\,. \tag{6}$$

Applying CG to (6) is more efficient compared to (2) because the spectrum of $\mathbf{M}^{-1}\mathbf{P}^{T}\mathbf{A}$ does not contain λ_1 and λ_2 (Fig. 3). $\mathbf{M}^{-1}\mathbf{P}^{T}\mathbf{A}$ has two zero eigenvalues indicating the rank deficiency of the projected system. CG is still applicable

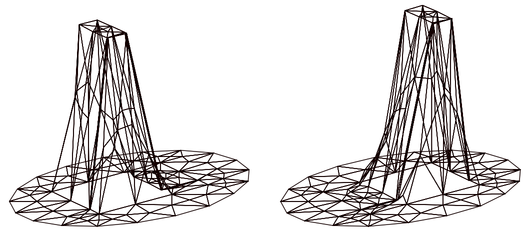

Fig. 5: Basis for the approximative eigenspace corresponding to the small eigenvalues of $\mathbf{M}^{-1}\mathbf{A}$.

as long as the solution **x** is restricted properly to $\mathbf{P}\mathbf{x}$ [4]. The deflated version of the ICCG solver (DICCG) provides an extra gain of convergence to which the extra work introduced by **P** in the algorithm is negligible (Fig. 2, Table I).

III. Approximative Eigenvectors

The exact determination of the considered eigenvectors would cost more work than the solution of (2). Fig. 4 indicates that approximations for these vectors are easily obtained on a heuristic basis. Approximative eigenvectors are constructed assuming the magnetic flux to be homogeneously distributed in the flux tubes formed by the high permeable iron parts and the air gaps of the model (Fig. 5). The performance of the deflated method depends on the accuracy of the determination of the eigenvectors corresponding to the crucial eigenmodes. The numerical tests (D^{*}ICCG in Fig. 2, Table I), however, show that this very easy approximation is sufficient to obtain a significant improvement. In practice, this kind of modellisation is always carried out before proceeding to a finite element model. The improved version of ICCG, presented here, recycles that information.

IV. Conclusions

The bad convergence properties of the incomplete Cholesky preconditioned Conjugate Gradient iterative method applied to a magnetostatic finite element model with large variations in reluctivities is overcome by supplying heuristic approximations for some important eigenvectors of the model to a deflated version of the solver.

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